Journal of Structural Geology, Vol. 12, No. 7, pp. 929 to 931, 1990
Printed in Great Britain

0191-8141/90 $03.00+0.00

Direction of shear

DONAL M. RAGAN

Department of Geology, Arizona State University, Tempe, AZ 85287-1404, U.S.A.

(*Received* 11 *October* 1989; *accepted in revised form* 12 *March* 1990)

Abstract—The direction of the maximum shear stress on any plane can be determined graphically from the traction vector. The orientation and magnitude of this vector are found from a fundamental property of symmetric tensors of second rank. Exactly the same method can be used to find the direction of the shear strain associated with any line.

INTRODUCTION

LISLE (1989), Means (1989) and De Paor (1990, this issue) describe methods for determining the direction of the maximum shear stress on a generally-oriented plane. There is yet another way which has some important conceptual and practical advantages. Elements of this method are scattered in a number of places in the engineering literature, for example, Jaeger & Cook (1979, pp. 30–32). The treatment by Goodman (1963) is perhaps the most complete, but it is unnecessarily complicated and there are a number of misprints in both text and illustrations.

The fundamental relationship between the components of the traction vector **T** (also called the stress vector) and the components of the unit vector **n** normal to a specified plane is,

$$T_i = \sigma_{ij} n_j, \tag{1}$$

where the σ_{ij} are components of the stress tensor (see Nye 1985, p. 82f). For computing and plotting it is always easier when the tensor is expressed in diagonal form. The components of the traction vector are then,

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}. \tag{2}$$

Performing this multiplication gives expressions for the components of **T** in the three co-ordinate direction,

$$T_1 = n_1\sigma_1, \quad T_2 = n_2\sigma_2, \quad T_3 = n_3\sigma_3 \tag{3}$$

and the magnitude of T is,

$$T = \sqrt{T_1^2 + T_2^2 + T_3^2}. \tag{4}$$

The direction cosines of T are found by dividing each component by this magnitude. With these, the corresponding direction angles are then,

$$\begin{aligned} \alpha_1 &= \arccos(T_1/T), \\ \alpha_2 &= \arccos(T_2/T), \\ \alpha_3 &= \arccos(T_3/T). \end{aligned} \tag{5}$$

The plane containing **T** and **n** intersects the specified plane to fix the direction of the shear.

PROCEDURE

To illustrate the technique, the solution of the example problem given by Means (1989) will illustrate the method. The principal stress direction σ_1 has a plunge and trend of 78°/337° and σ_3 has a plunge and trend of 11°/132° (see Fig. 1a). If σ_1 = 50 MPa, σ_2 = 30 MPa and σ_3 = 20 MPa, what is the direction of the shearing component of the traction acting on the plane whose pole has a plunge and trend of 30°/250°?

There are two closely related approaches to solving this problem graphically: (1) the attitudes may be retained in the original geographical co-ordinates, or (2) the principal stresses may be rotated into net co-ordinates. The first avoids the rotation, but requires the construction of small circles about inclined axes (Ragan 1985, pp. 304–306), while the second utilizes the small circles printed on the net. There is not much difference in the amount of work involved, and the choice depends on one's facility with the plotting techniques. I prefer the second because it seems to display the elements of the problem more simply, but the basic method works for both.

Adopting the convenient set of axes $+x_1$ down, $+x_2$ north and $+x_3$ east, the principal stress directions are rotated into coincidence with these (Fig. 1b). In this frame, the attitude of the pole of the plane of interest is 30°/201°.

The direction cosines of **n** can be obtained in either of two ways. The direction angles of the pole can be measured from the points representing the co-ordinate directions, or the plunge p and trend t of the pole can be converted using the formulae (cf. Ragan 1985, p. 365),

$$n_1 = \sin p, \quad n_2 = \cos p \cos t, \quad n_3 = \cos p \sin t. \tag{6}$$

Construction

(1) On the lower-hemisphere of a stereogram plot **n** representing the pole of the plane of interest, and add the great circle trace of the plane itself.

(2) Convert the plunge and trend of **n** to direction cosines using equation (6), with the results,

$$n_1 = 0.50000, \quad n_2 = -0.80850, \quad n_3 = -0.31036.$$

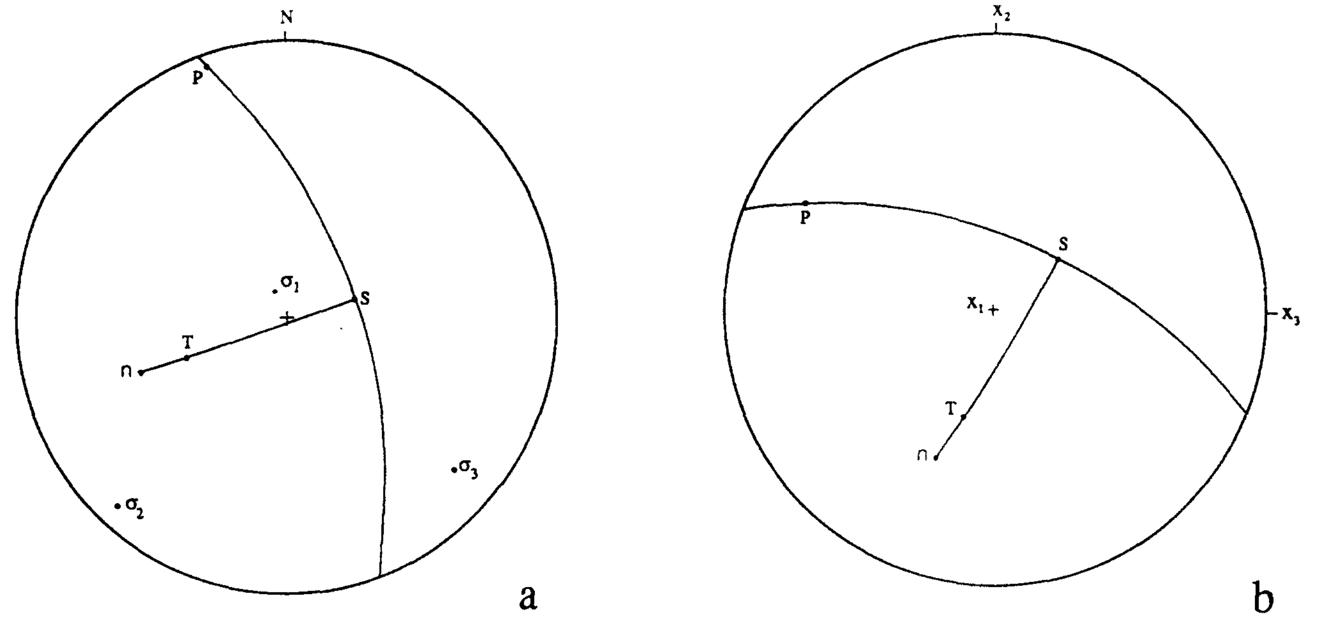

Fig. 1. The shearing component of a traction on a generally-oriented plane. **n** = normal to the plane, **T** = traction vector, S = shear direction on the plane and **P** = the direction of $\tau = 0$ on the plane: (a) elements of the original problem (after Means 1989); (b) solution using rotation.

(3) With equation (3) determine the components,

$$T_1 = 25.00, \quad T_2 = -24.26, \quad T_3 = -6.21.$$

Then from equation (4), the magnitude of the traction vector is $T = 35.38$.

(4) With equation (5), the direction angles of **T** are then,

$$\alpha_1 = 45.0°, \quad \alpha_2 = -46.7°, \quad \alpha_3 = -79.9°.$$

(5) Locate **T** at the intersections of small circles with radii equal to these direction angles drawn about each point representing a co-ordinate axis.

(6) The great circle through **T** and **n** intersects the trace of the plane at the shear direction **S**, with an attitude of 56°/053°.

After rotating the elements of the problem back to their original orientations, this result is essentially the same as obtained by Means (1989).

EXTENSIONS

This simple analytical–graphical method may now be extended to other situations or to give additional useful information.

(1) The magnitudes of the normal and shear components σ and τ are easily obtained from,

$$\sigma = T\cos\theta \quad \text{and} \quad \tau = T\sin\theta, \tag{7}$$

where θ is the angle between **n** and **T**.

(2) The shear direction may be calculated in two steps:

(a) pole **P** of the plane containing the normal and traction vectors is found from the cross product $\mathbf{P} = \mathbf{n} \times \mathbf{T}$, and this also locates the direction of $\tau = 0$ on the plane;

(b) shear direction **S** is then obtained from the cross product $\mathbf{S} = \mathbf{n} \times \mathbf{P}$.

Another application of this method is to determine the direction of finite shear strain associated with a line in the deformed state. In diagonal form, the controlling equation is,

$$\begin{bmatrix} \lambda'_1 l'_1 \\ \lambda'_2 l'_2 \\ \lambda'_3 l'_3 \end{bmatrix} = \begin{bmatrix} \lambda'_1 & 0 & 0 \\ 0 & \lambda'_2 & 0 \\ 0 & 0 & \lambda'_3 \end{bmatrix} \begin{bmatrix} l'_1 \\ l'_2 \\ l'_3 \end{bmatrix} \tag{8}$$

where λ'_1, λ'_2 and λ'_3 are the principal reciprocal quadratic elongations, and (l'_1, l'_2, l'_3) are the direction cosines of a line of interest. Normalizing the components in the resulting column matrix then gives the direction cosines of the normal to the strain ellipsoid. The plane containing the line and the derived normal to the strain ellipsoid contains the shear direction, and the angle between these two directions is the angle of shear.

In these applications a plot of the basic geometrical data needs only a single additional point to determine the shear direction. This point is obtained by a few simple calculations. With this result, the analysis may then be extended to a determination of several additional parameters, and part or all of the process may be programmed for a computer solution.

This method exploits a fundamental property common to all symmetric second-rank tensors, including the stress, finite strain, infinitesimal strain and strain-rate tensors. For all of these the calculation yields the orientation of the *radius-normal* from the *radius vector* and the principal values. Nye (1985, p. 28) summarizes the geometric relationships between these two directions:

> If $p_i = S_{ij}q_j$, the direction of **p** for a given **q** may be found by first drawing, parallel to **q** a radius

vector OP of the representation quadric, and then taking the normal to the quadric at P.

This construction does not depend on the representation quadric of the tensor being an ellipsoid (see Nye 1985, pp. 28–30, for the treatment of hyperboloids and the case where the radius vector does not intersect the quadric at a real point).

Acknowledgements—I thank Chris Sanders for reading an early draft, and Richard Lisle, Declan De Paor and especially Win Means for their reviews and comments.

REFERENCES

De Paor, D. G. 1990. The theory of shear stress and shear strain on planes inclined to the principal directions. *J. Struct. Geol.* **12**, 923–927 (this issue).

Goodman, R. E. 1963. The resolution of stresses in rock using stereographic projection. *Int. J. Rock Mech. Min. Sci.* **1**, 93–103.

Jaeger, J. C. & Cook, N. G. W. 1979. *Fundamentals of Rock Mechanics* (3rd edn). Chapman & Hall, London.

Lisle, R. J. 1989. A simple construction for shear stress. *J. Struct. Geol.* **11**, 493–495.

Means, W. D. 1989. A construction for shear stress on a generally-oriented plane. *J. Struct. Geol.* **11**, 625–627.

Nye, J. F. 1985. *Physical Properties of Crystals* (rev. edn). Oxford University Press, London.

Ragan, D. M. 1985. *Structural Geology, an Introduction to Geometrical Techniques* (3rd edn). John Wiley, New York.